ON THE SPHERICAL REPRESENTATION OF A SURFACE

By Paul Saurel.

The focal lines corresponding to a given point on a surface are two straight lines each of which is drawn through one of the principal centres of curvature, parallel to the line of curvature that corresponds to the other principal centre. And it is well known that all the normals to the surface drawn from the neighborhood of the given point intersect these two lines.* In this note I should like to indicate how this property of the focal lines enables us to state in geometric terms a very elegant demonstration, due to Stäckel,† of a fundamental relation in the theory of the spherical representation of a surface.

Let P and Q be neighboring points on a surface and let the projections of PQ on the axes be dx, dy, dz. If P' and Q' be the points in which the lines through the origin parallel to the normals at P and Q pierce the surface of the unit sphere whose centre is at the origin, and if X, Y, Z be the direction cosines of the normal at P, the projections of $P'Q'$ on the axes will be dX, dY, dZ.

From the point F_1, in which the normal through Q cuts the focal line parallel to the first line of curvature through P, draw a line parallel to the normal at P; this line will cut the first line of curvature in a point P_1. Since the focal line on which F_1 lies passes through the second centre of curvature, the distance F_1P_1 will be equal to the second radius of curvature R_2. Moreover, the line QP_1 will be parallel to $Q'P'$ and its projections on the axes will be equal, in magnitude and in sign, to R_2dX, R_2dY, R_2dZ. It follows that the projections of PP_1, the geometric sum of PQ and QP_1, are equal to

$$dx + R_2dX, \quad dy + R_2dY, \quad dz + R_2dZ. \tag{1}$$

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* Cf. G. Scheffers, *Einführung in die Theorie der Flächen*, p. 166.

† P. Stäckel, *Bulletin des sciences mathématiques*, 2nd series, vol. 27, p. 139; 1903.

In like manner, through the point F_2 in which the normal at Q cuts the second focal line, draw a line parallel to the normal at P; this line will cut the second line of curvature in a point P_2. If we denote the first radius of curvature by R_1, the projections of QP_2 will be R_1dX, R_1dY, R_1dZ, and those of PP_2

$$dx + R_1dX, \quad dy + R_1dY, \quad dz + R_1dZ. \tag{2}$$

If we remember that the lines of curvature are at right angles to each other, we get at once from (1) and (2) the equation

$$\sum (dx + R_2dX)(dx + R_1dX) = 0, \tag{3}$$

and from this we obtain without difficulty the fundamental relation

$$\frac{1}{R_1R_2}\sum dx^2 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\sum dx dX + \sum dX^2 = 0. \tag{4}$$

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